An elementary development of mass–energy equivalence

Daniel J. Steck
Department of Physics, Saint John's University, Collegeville, Minnesota 56321
Frank Rioux
Department of Chemistry, Saint John's University, Collegeville, Minnesota 56321

(Received 29 December 1980; accepted for publication 17 May 1982)

In this note we describe a simple derivation of the mass–energy equivalence equation that we have not seen previously in the literature. The derivation is of the type which Einstein used[1,2] and employs a "thought experiment" in which an elementary photon emission event is examined by two sets of inertial observers. A comparison of the analyses of these observers brings the validity of the principle of momentum conservation into question. However, since the conservation of momentum is the cornerstone of dynamics it is preserved at all costs. This is accomplished by allowing the concepts of the mass and energy to undergo redefintion.

A block which is stationary with respect to observers A^0 and B^0 emits two photons of equal frequency in opposite directions as shown in Fig. 1. Using the principle of the conservation of momentum, and the Maxwell ($|p| = E/c$) and Planck–Einstein ($E = h\nu$) relationships for electromagnetic radiation, A^0 and B^0 infer that the change in momentum of the block is zero. In other words,

$$\Delta p^0_{\text{block}} = -\Delta p^0_{\text{photon}} = -(\nu^0_B - \nu^0_A)\,h/c = 0\,, \tag{1}$$

since $\nu^0_A = \nu^0_B = \nu^0$. In addition these observers find that the energy change of the block is

$$\Delta E^0 = -2h\nu^0\,. \tag{2}$$

Relative to observers A and B, shown in Fig. 2, the block is moving with velocity v. Their statement of the conservation of momentum is

$$\Delta p_{\text{block}} + (\nu_B - \nu_A)\,h/c = 0\,, \tag{3}$$

where for these observers $\nu_A \neq \nu_B \neq \nu^0$.

Owing to the Doppler effect, B finds that photon B, emitted by a source approaching with veloicty v, has a frequency given by

$$\nu_B = [(1 + v/c)/(1 - v/c)]^{1/2}\,\nu^0\,. \tag{4}$$

Observer A, on the other hand, finds that photon A, emitted by a source receding with velocity v, has a frequency given by

$$\nu_A = [(1 - v/c)/(1 + v/c)]^{1/2}\,\nu^0\,. \tag{5}$$

Substitution of Eqs. (4) and (5) into Eq. (3) yields after rearrangement

$$\Delta p_{\text{block}} + [2h\nu^0/(1 - v^2/c^2)^{1/2}](v/c^2) = 0\,. \tag{6}$$

These observers find that the energy change of the block to be

$$\begin{aligned}\Delta E &= -h\nu_A - h\nu_B\\ &= -2h\nu^0/(1 - v^2/c^2)^{1/2}\\ &= \Delta E^0/(1 - v^2/c^2)^{1/2}\,.\end{aligned} \tag{7}$$

Substitution of Eq. (7) into Eq. (6) gives

$$\Delta p_{\text{block}} - \Delta E\,(v/c^2) = 0\,. \tag{8}$$

The classical definition of momentum is maintained by requiring that $p_{\text{block}} = mv$. Since no recoil is observed in the rest frame, the relativity principle requires that $\Delta v = 0$ in the moving frame also. This means that the principle of the conservation of linear momentum can only be preserved for A and B by postulating a mass change for the block upon emission of the photons, or that $\Delta p_{\text{block}} = \Delta(mv) = v\Delta m$. In light of these considerations, Eq. (8) can be written as

$$\Delta E = \Delta mc^2\,. \tag{9}$$

We note that an equivalent equation ($\Delta E^0 = \Delta m^0c^2$) can be written for the rest frame by combining Eqs. (7) and (9) and interpreting $\Delta m(1 - v^2/c^2)^{1/2}$ as the change in rest mass of the block resulting from the emission of the two photons:

$$\Delta m^0 = \Delta m(1 - v^2/c^2)^{1/2}\,. \tag{10}$$

Generalizing on the basis of Eqs. (9) and (10) suggests that

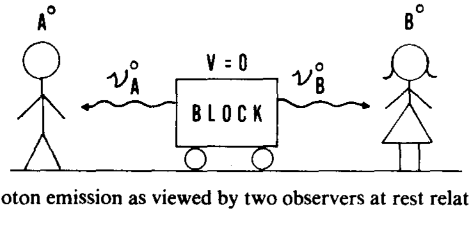

Fig. 1. Photon emission as viewed by two observers at rest relative to the block.

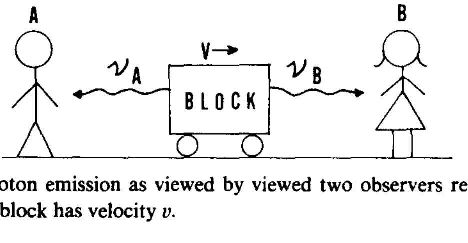

Fig. 2. Photon emission as viewed by viewed two observers relative to which the block has velocity v.

we can write

$$E = mc^2 , \tag{11}$$

where

$$m = m^0(1 - v^2/c^2)^{-1/2} . \tag{12}$$

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[1] A. Einstein, Ann. Phys. **18**, 639 (1905). An English translation can be found in *The Principle of Relativity*, edited by H. A. Lorentz *et al.* (Dover, New York, 1952), p. 69–71.

[2] A. Einstein, in *Out of My Later Years* (Philosophical Library, New York, 1950), p. 116.