Proof of Parallel Axis Theorem: Consider an object of arbitrary shape, as shown in the drawing. We choose a coordinate system in which the origin \((x = 0, y = 0)\) is at the center of mass of the object. We consider first an axis of rotation through point A and perpendicular to the paper. This point A is at coordinates \(x = a, y = b\). We calculate the moment of inertia about point A by considering a sum over all of the mass points that make up the object. (We really should do an integral, but this approach should keep things a little simpler.) Consider, for example, the mass point \(m_i\) located at \(x = x_i, y = y_i\). The moment of inertia of object about the axis through A is just the sum over all such mass points:

\[
I_A = \sum_i m_i [(x_i - a)^2 + (y_i - b)^2]
\]

Take a moment to be sure you understand this equation. Then, expand the terms in parentheses and collect terms, as follows as follows:

\[
I_A = \sum_i m_i [x_i^2 - 2ax_i + a^2 + y_i^2 - 2by_i + b^2]
\]

\[
= \sum_i m_i (x_i^2 + y_i^2) - 2a \sum_i m_i x_i - 2b \sum_i m_i y_i + (a^2 + b^2) \sum_i m_i
\]

Consider each of these terms in turn. Note that the first term is the moment of inertia about the origin—in other words, the moment of inertia about a parallel axis through the center of mass:

\[
I_{cm} = \sum_i m_i (x_i^2 + y_i^2)
\]

The next two terms are related to the x and y components of the center of mass. Thus,

\[
x_{cm} = \frac{1}{M} \sum_i m_i x_i = 0
\]

where \(M\) is the total mass of the object (\(M = \sum m_i\)). (Since we have chosen the center of mass at the origin, the coordinates of the center of mass are \(0,0\).) A similar equation holds for the y component of the center of mass. Hence the second and third terms in the expression for \(I_A\) are zero:

\[
\sum_i m_i x_i = 0 \quad \text{and} \quad \sum_i m_i y_i = 0.
\]

The last term is simply

\[
(a^2 + b^2) \sum_i m_i = M(a^2 + b^2)
\]

Hence Equation (1), our expression for \(I_A\) becomes

\[
I_A = I_{cm} + M(a^2 + b^2) = I_{cm} + Mh^2
\]

where \(h = \sqrt{a^2 + b^2}\) is just the perpendicular distance from an axis through the center of mass to an axis through A. Hence we have proven the parallel axis theorem: QED!