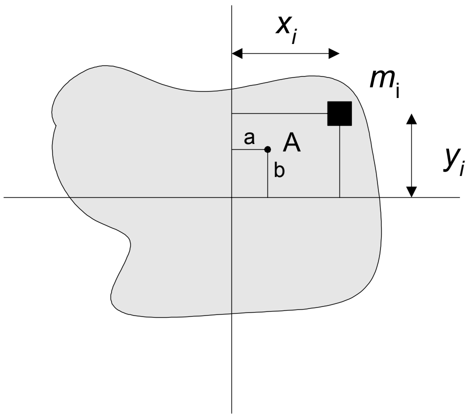
Proof of Parallel Axis Theorem: Consider an object of arbitrary shape, as shown in the drawing.



We choose a coordinate system in which the origin (x = 0, y = 0) is at the center of mass of the object. We consider first an axis of rotation through point A and perpendicular to the paper. This point A is at coordinates x = a, y = b. We calculate the moment of inertial about point A by considering a sum over all of the mass points that make up the object. (We really should do an integral, but this approach should keep things a little simpler.) Consider, for example, the mass point m_i located at $x = x_i$, $y = y_i$. The moment of inertial of object about the axis through A is just the sum over all such mass points:

$$I_A = \sum_i m_i\left[(x_i - a)^2 + (y_i - b)^2\right]$$

Take a moment to be sure you understand this equation. Then, expand the terms in parentheses and collect terms, as follows as follows:

$$\begin{aligned} I_A &= \sum_i m_i\left[x_i^2 - 2ax_i + a^2 + y_i^2 - 2by_i + b^2\right] \\ &= \sum_i m_i(x_i^2 + y_i^2) - 2a\sum_i m_i x_i - 2b\sum_i m_i y_i + (a^2 + b^2)\sum_i m_i \end{aligned} \qquad (1)$$

Consider each of these terms in turn. Note that the first term is the moment of inertia about the origin—in other words, the moment of inertia about a parallel axis through the center of mass:

$$I_{cm} = \sum_i m_i(x_i^2 + y_i^2)$$

The next two terms are related to the x and y components of the center of mass. Thus,

$$x_{cm} = \frac{1}{M}\sum_i m_i x_i = 0$$

where M is the total mass of the object ($M = \sum_i m_i$). (Since we have chosen the center of mass at the origin, the coordinates of the center of mass are 0,0.) A similar equation holds for the y component of the center of mass. Hence the second and third terms in the expression for I_A are zero:

$$\sum_i m_i x_i = 0 \quad \text{and} \quad \sum_i m_i y_i = 0.$$

The last term is simply

$$(a^2 + b^2)\sum_i m_i = M(a^2 + b^2)$$

Hence Equation (1), our expression for I_A becomes

$$I_A = I_{cm} + M(a^2 + b^2) = I_{cm} + Mh^2$$

where $h = \sqrt{a^2 + b^2}$ is just the perpendicular distance from an axis through the center of mass to an axis through A. Hence we have proven the parallel axis theorem: QED!