

## Work and Potential Energy

We start with the idea that the work done by a **conservative force** depends only on the initial and final positions of the mass in question. For example (see YF Fig. 7.2(a) and Eq. (7.1)). If we drop a mass  $m$  through a height  $h = y_1 - y_2$  (where  $y_1 > y_2$ ), then the positive work done by gravity is

$$W = mgh = mg(y_1 - y_2) = mgy_1 - mgy_2. \quad (1)$$

We get the same result if the same mass slides without friction along an inclined plane, as long as the vertical change in height is the same. Remember that we proved this result in Question 6.8. Therefore the work done meets our definition of a conservative force.

For any conservative force, it is possible to define a new quantity, the potential energy  $U$ , where  $U$  is some function of the coordinates that describe the position of the mass. If  $W$  is the work done, then

$$W = \int_i^f \vec{F} \cdot d\vec{s} \equiv U_i - U_f.$$

In words, the area under the graph of force vs. position (the integral in the last equation) is the difference between the initial and final values of  $U$ . It is often convenient to rewrite this equation as

$$W = U_i - U_f = -\Delta U = -(U_f - U_i). \quad (2)$$

Eq. (2) is important; be sure to keep it in mind! Note that Eq. (1) tells us that the work done is equal to the negative of the change in potential energy: **Thus if work is positive, the mass loses potential energy**, and vice-versa.

We now call on the **work-energy theorem**, which says

$$KE_f - KE_i = W \quad (3)$$

where KE is the kinetic energy,  $\frac{1}{2}Mv^2$ .

Next, we combine Equations (2) and (3):

$$KE_f - KE_i = W = -(U_f - U_i) \quad (4)$$

which after a little algebra, can be written in the form:

$$KE_f + U_f = KE_i + U_i.$$

In other words, the sum of kinetic energy and potential energy add up to the same value, no matter where the mass is at any given instant. The total mechanical energy is a constant, or, as we often say, is conserved.

We find the form of the potential energy function  $U$  by inspecting the work done by a particular force.

**Example 1: Force of gravity near the earth's surface.** For a mass falling through a constant gravitational force near the surface of the earth, we found that Eq. (1) gives the

work done by gravity as the mass falls from  $y_1$  to  $y_2$ . This result suggests that we define gravitational potential energy function by

$$U_{grav}(y) = Mgy$$

so that for gravitational forces, we have from Eq. (4) that

$$\begin{aligned} KE_f - KE_i &= W = -(Mgy_f - Mgy_i) \quad \text{or} \\ KE_f + Mgy_f &= KE_i + Mgy_i \end{aligned}$$

Take a moment and persuade yourself that this equation is consistent with Eq.(1).

**Example 2: Spring force.** We have found that the work done by a spring as it moves from some initial position  $x_i$  to some final position  $x_f$  is given by

$$\begin{aligned} W &= \int_{x_i}^{x_f} -kx \, dx = -\frac{1}{2}kx^2 \Big|_{x_i}^{x_f} \\ &= -\frac{1}{2}kx_f^2 + \frac{1}{2}kx_i^2. \end{aligned}$$

Note that the signs are different from YF's Eq. (6.10), because they calculate the work done **on** the spring, not by the spring—see the CAUTION on p. 194. Note that YF clear up this sign dilemma in Chapter 7, on page 223.

This result for the work suggests the following definition for the spring potential energy:

$$U_{spring}(x) = \frac{1}{2}kx^2.$$

Now, recalling that  $W = -\Delta U = -\frac{1}{2}kx_f^2 + \frac{1}{2}kx_i^2$ , we have from Eq. (4) that

$$\begin{aligned} KE_f - KE_i &= W = -\frac{1}{2}kx_f^2 + \frac{1}{2}kx_i^2 \quad \text{or} \\ KE_f + \frac{1}{2}kx_f^2 &= KE_i + \frac{1}{2}kx_i^2. \end{aligned}$$