Otto Cycle
Calculation of entropy change
(following Baierlein)

Consider the constant volume path $4 \rightarrow 1$, using Baierlein's notation in Figure 3.3. We imagine a reservoir at temperature $T_1 = T_{hot}$ placed in thermal contact with the engine (a piston/cylinder arrangement containing an ideal gas).

The reservoir loses heat at constant temperature, so its entropy change is

$$\Delta S_{res} = -\frac{Q_{in}}{T_1} = -\frac{C_v(T_1 - T_4)}{T_1} < 0 .$$

Note, of course, that $Q_{in} = C_v(T_1 - T_4)$ is just the thermal energy needed to raise the temperature of the gas from T_4 to T_1 at constant volume.

The entropy of the engine is, as we have seen,

$$\Delta S_{eng} = \int \frac{dQ}{T} = C_v \int_{T_4}^{T_1} \frac{dT}{T}$$

$$= C_v \ln \frac{T_1}{T_4} > 0.$$

We expect the total entropy change to be positive, since thermal energy is being transferred across a finite temperature difference. So we need to investigate

$\Delta S_{total} = \Delta S_{res} + \Delta S_{engine}$, or

$$\Delta S_{total} = C_v \ln \frac{T_1}{T_4} - \frac{C_v(T_1 - T_4)}{T_1}$$

To investigate the sign of this quantity, note that $T_1 = T_4 + T_1 - T_4$, so that

$$\ln \frac{T_1}{T_4} = \ln\left(1 + \frac{T_1 - T_4}{T_4}\right) = \ln(1 + x),$$

where $x \equiv \frac{T_1 - T_4}{T_4}$. Notice that since $T_1 > T_4$, we have $0 < x < 1$. Note also that in terms of x, the entropy change of the reservoir is

$$\Delta S_{res} = -\frac{C_v(T_1 - T_4)}{T_1} = -C_v \frac{T_1 - T_4}{T_4} \frac{T_4}{T_1}$$

$$= -C_v \frac{x}{1 + x}$$

where I leave it as an exercise to show that $T_4 / T_1 = 1/(1 + x)$. In terms of x, the total entropy change is therefore

$$\Delta S_{total} = C_v \left[\ln(1+x) - \frac{x}{1+x} \right].$$

If we expand $\ln(1+x)$ in a Taylor series (try it, or get *Mathematica* or *Mathcad* to do it), we obtain

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots ,$$

and substituting

$$\Delta S_{total} = C_v \left[\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \right) - \frac{x}{1+x} \right].$$

We can if we wish do a numerical investigation: If one defines this function of x in *Mathematica*, one finds that ΔS_{total} is positive for any x in the range $0 < x < 1$.

But we can also give an analytic proof. First, reorder the terms as follows:

$$\Delta S_{total} = C_v \left[\left(x - \frac{x}{1+x} - \frac{x^2}{2} \right) + \left(\frac{x^3}{3} - \frac{x^4}{4} \right) + \ldots \right].$$

Note that $x - \frac{x}{1+x} = \frac{x^2}{1+x}$, so that the total entropy change becomes

$$\Delta S_{total} = C_v \left[\left(\frac{x^2}{1+x} - \frac{x^2}{2} \right) + \left(\frac{x^3}{3} - \frac{x^4}{4} \right) + \ldots \right]$$

If we remember that $0 < x < 1$ (recall the definition of x), it is apparent by inspection that each of the terms in the parentheses ( ) is positive, and so $\Delta S_{total} > 0$.

The calculation for ΔS_{total} along the path $2 \to 3$ is also positive, and goes through in the way I worked it out in class.