On classifying finite edge colored graphs with two transitive automorphism groups

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Abstract

This paper classifies all finite edge colored graphs with doubly transitive automorphism groups. This result generalizes the classification of doubly transitive balanced incomplete block designs with \( \lambda = 1 \) and doubly transitive one-factorizations of complete graphs. It also provides a classification of all doubly transitive symmetric association schemes.

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The classification of finite simple groups in the 1980s has led to theorems classifying a variety of designs and geometric structures. Edge colored graphs generalize balanced incomplete block designs with \( \lambda = 1 \) and one-factorizations of complete graphs. This paper classifies the doubly transitive edge colored graphs (abbreviated 2-t ec-graphs), extending results of Kantor [14] and Cameron and Korchmaros [8]. The 2-t symmetric graph designs of Cameron [7] when \( \lambda = 1 \) match the 2-t ec-graphs for which the number of colors equals the number of vertices. Edge colored graphs are closely related to the rainbows in Aschbacher [2].

Definitions. An edge colored graph \((V, C)\) is a finite set \(V\) of vertices and a function \(C\) from the set \(E\) of all undirected edges \(ab\), where \(a \neq b\), onto a non-empty set \(C(E)\) of edge colors. We assume that \(|V| \geq 2\), where \(|V|\) is the number of elements in \(V\). An automorphism \(\alpha\) of \((V, C)\) is a bijection of \(V\) such that for all edges \(ab\) and \(cd\), \(C(ab) = C(cd)\) if and only if \(C(\alpha(a)\alpha(b)) = C(\alpha(c)\alpha(d))\). An edge colored graph \((V, C)\) is doubly transitive if and only if its group of automorphisms \(\text{Aut}(V, C)\) is doubly transitive (2-t) on \(V\).

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Example 1. For any $V$ the 2-t ec-graph obtained by setting $C_M(ab) = 1$ for all edges $ab$ is called the monochromatic ec-graph $(V, C_M)$. The 2-t ec-graph obtained by setting $C_T(ab) = ab$ is called the trivial ec-graph $(V, C_T)$. Then $A(V, C_M) = A(V, C_T) = S_V$, the symmetric group on $V$.

Example 2. In a balanced incomplete block designs (BIBD) with $\lambda = 1$, vertex set $V$, and $B$ the set of blocks, denote by $B(a, b)$ the unique block (line) containing $a$ and $b$. The edge colored graph $(V, C_B)$ is derived from this BIBD if $C_B(ab) = B(a, b)$. Kantor [14] classified all finite 2-t BIBDs with $\lambda = 1$, including affine spaces $AG(n, p^k)$ and projective spaces $PG(n, p^k)$ over the field of order $p^k$.

Example 3. A one-factorization is an ec-graph where the edges of each color determine a regular graph of degree one. Cameron and Korchmaros [8] classified all finite 2-t one-factorizations and Cameron [6] classified the triply transitive (3-t) ones.

Theorem 1 below classifies the ec-graphs whose automorphism groups are 3-t.

**Theorem 1.** If $(V, C)$ is a finite doubly transitive edge colored graph and $G$ is a group acting triply transitively on $V$ with $G \leq A(V, C)$, then $(V, C)$ is monochromatic or trivial with $|V| \geq 2$ or

(i) the doubly transitive one-factorization based on the affine space $AG(n, 2)$, where parallel edges are the same color and $|V| = 2^n$, or

(ii) the doubly transitive one-factorization in Fig. 1 and $|V| = 6$.

**Proof.** The case $|V| = 2$ is obvious. For $|V| \geq 3$ suppose first that there are adjacent edges $ab$ and $ax$ such that $C(ab) = C(ax)$. By 3-t for each $y \in V$ distinct from $a$ and $b$, there is an automorphism fixing $a$ and $b$ and moving $x$ to $y$. So for all $y \neq a$, $C(ay) = C(ab)$. In turn, for all $z \neq y$, $C(yz) = C(ya) = C(ab)$, and $(V, C)$ is monochromatic. We may thus assume that adjacent edges are different colors and there are distinct $a$,
Example 4. If $V$ is a two-dimensional vector space over a field $F$, $\langle \cdot, \cdot \rangle$ is a non-degenerate, symmetric bilinear form on $V$, and $C(ab) = \langle a - b, a - b \rangle$, then $(V, C)$ is a 2-t ec-graph. This construction does not generalize to higher dimensional finite spaces because of isotropic elements (see [4]). Any metric space $(X, d)$ becomes an ec-graph by setting $C(ab) = d(a, b)$, and conversely any finite ec-graph becomes a metric space by assigning numbers in $[1, 2]$ to the colors.

The first two sections of this paper classify 2-t ec-graphs based on their groups of automorphisms. From the classification of finite simple groups, the finite 2-t groups split into two large collections and one other family. The first collection consists of groups with a 2-t simple subgroup. The 2-t subgroups of some affine group form the second collection. The remaining family consists of groups containing $PGL(2, 8) = 2G_2(3)$ acting on a set with 28 elements (see [14]). Theorem 6, the main result of Section 1, classifies all 2-t ec-graphs whose groups of automorphisms contain some 2-t simple group. In essence, these 2-t ec-graphs are found in Examples 1–3 above and 5–11 below. Theorem 7, which closes Section 1, classifies the 2-t ec-graphs whose automorphism groups contain $PGL(2, 8)$. Section 2 classifies, to the extent practical, the 2-t ec-graphs whose groups of automorphisms are 2-t subgroups of some affine group. This collection of graphs, which includes those of Example 4, has a far more extensive and complicated structure than those in Section 1, making an explicit counterpart to Theorems 6 and 7 infeasible. Examples 12 and 13 give general constructions for all such 2-t ec-graphs. Section 3 classifies regular 2-t ec-graphs, as in Examples 3 and 4, where the edges of each color form a regular graph on $V$. It also classifies 2-t point color symmetric graphs and 2-t symmetric association schemes.

1. Non-affine automorphism groups

Here we consider only 2-t ec-graphs whose groups of automorphisms contain a finite simple group or $PGL(2, 8)$ acting on a set of 28 elements. We present the remaining examples of 2-t ec-graphs with non-affine automorphism groups and lemmas providing the means of determining all of the possibilities. Lemma 2 matches possible 2-t ec-graphs with appropriate subgroups of a 2-t group. Lemma 3 lets us use only the finite simple groups and $PGL(2, 8)$, rather than all groups containing them.

Example 5. For $V = PG(n, 2)$, each line has three points incident with it. Define $C(E) = V$ and $C(ab) = e$ iff $e \in B(a, b)$, $c \neq a$ and $c \neq b$ (see Fig 2). Then
\(A(V, C) = PGL(n + 1, 2)\) and \((V, C)\) is a 2-t ec-graph as well as a 2-t symmetric graph design with \(\lambda = 1\) of Cameron [7]. This construction also applies to the 15-point ec-graph with \(A(V, C) = \mathcal{A}_7\).

**Definition.** For edge colored graphs \((V, C)\) and \((V, C')\), \((V, C)\) is weaker than \((V, C')\), written \((V, C) \leq (V, C')\), iff there is a surjection \(\gamma : C(E) \rightarrow C'(E)\) such that for all \(ab, \gamma(C(ab)) = C'(ab)\).

**Example 6.** Let \(V = PG(n, 3), B\) its set of lines (blocks), and \((V, C_B)\) the 2-t ec-graph derived from this BIBD with \(\lambda = 1\). We define a weaker 2-t ec-graph \((V, C)\) by splitting the six same-colored edges from each line into three pairs as in Fig. 3. More precisely, \(C(E) = B \times \{1, 2, 3\}\) and for each line \(l\) in \(B\) with any labeling \(l_i\) of its points, for \(i \in \{0, 1, 2, 3\}\), define \(C(l_0) = (l, i)\) and \(C(l_i) = (l, k)\), where \(i, j, k\) are distinct elements in \(\{1, 2, 3\}\). Then \(A(V, C) = A(V, C_B)\) and \((V, C)\) is a 2-t ec-graph.

**Definition.** Two edge colored graphs \((V, C)\) and \((V', C')\) are isomorphic iff there are bijections \(\beta : V \rightarrow V'\) and \(\gamma : C(E) \rightarrow C'(E')\) such that for all \(a\) and \(b\), \(\gamma(C(ab)) = C'(\beta(a)\beta(b))\).

![Diagram](image.png)

**Fig. 2.** The vertex \(c\) is on the lines on \(a\) and \(b\), on \(d\) and \(e\), and on \(f\) and \(g\). Thus \(C(ab) = C(de) = C(fg) = c\).

![Diagram](image.png)

**Fig. 3.**
Table 1

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**Example 7.** Let $V = PG(n, 5)$, $B$ its set of lines, and $(V, C_B)$ the 2-t ee-graph derived from this BIBD with $\lambda = 1$. Analogously to Example 6 we split each color of $C_B$ into five colors with each line colored in $(V, C)$ as in Fig. 1. More precisely, $C(E) = B \times \{1, 2, 3, 4, 5\}$. Fix a line $k$ and label its points $k_i$, where $i \in \{0, 1, 2, 3, 4, 5\}$. For each $l \in B$, there is $z \in PGL(n + 1, 5)$ mapping $k$ to $l$. Because $PGL(n + 1, 5)$ is 3-t on $l$ the choice of $z$ is immaterial up to isomorphism. Label the points of $l$ as $l_x = z(k_i)$. If we define $C(l_x) = (l, i \ast j)$, where $i \ast j$ is given by Table 1, $A(V, C) = A(V, C_B)$, and $(V, C)$ is a 2-t ee-graph.

**Example 8.** The constructions of Examples 6 and 7 apply to two BIBDs of unital over the field $Z_5$ and one BIBD over the field $Z_7$, yielding three more 2-t ee-graphs, whose automorphism groups are $U_3(3)$, $PGL(2, 8)$, and $U_1(5)$, respectively. Theorem 7 considers $PGL(2, 8)$, which is a not a simple group.

**Example 9.** Let $X$ be the $2n$-dimensional vector space over $Z_2$ and $G = PSp(2n, 2)$, for $n > 2$. We follow the notation in Dixon and Mortimer [11, pp. 245–248]. Now $G$ is 2-t on the subsets $\Omega^+$ and $\Omega^-$ of $X$. For $\Omega_i$ either $\Omega^+$ or $\Omega^-$ and $\theta_a, \theta_b \in \Omega'$ with $\theta_a \neq \theta_b$, define $C(\theta_a, \theta_b) = a + b$. The transvection switching $\theta_a$ and $\theta_b$ is in $\lambda(\Omega, C)$. Since the transvections generate $G$ both $(\Omega^+, C)$ and $(\Omega^-, C)$ are 2-t ee-graphs.

**Definitions.** For a color $c$, an $c$-chromomorphism $\kappa$ is an automorphism such that for every edge $ab$, if $C(ab) = c$, then $C(\kappa(a)\kappa(b)) = c$; that is, $\kappa$ preserves the color $c$, although not necessarily other colors. For a given $c$ the subgroup of all $c$-chromomorphisms is denoted $K(V, C, c)$, abbreviated $K(c)$. If we are focusing on the color of an edge $ab$, we write $K(V, C, C(ab))$ or $K(C(ab))$.

Lemma 2 below is a key to using the groups $K(c)$ to generate 2-t ee-graphs. Recall $G_{(a,b)}$ is the stabilizer in $G$ of the edge $ab$. In Lemma 2 the lattice of subgroups $K$ with $A(V, C)_{(a,b)} \leq K \leq A(V, C)$ gives a corresponding lattice of 2-t ee-graphs. The trivial and monochromatic colorings correspond to $A(V, C)_{(a,b)}$ and $A(V, C)$, respectively. Note that we may use any edge $ab$ because a 2-t group is transitive on the edges.

**Lemma 2.** Let $G$ be a doubly transitive group acting on a finite set $V$ and $ab$ any edge. Then for each subgroup $K$ such that $G_{(a,b)} \leq K \leq G$ there is, up to
orbit is $\{x\}$, which we assume. Then $(V, C_B) \leq (V, C)$, where $(V, C_B)$ is derived from the BIBD $PG(3, 2)$. Since $c_{ab} = 2$ in $(V, C)$ any two same colored lines are disjoint. Given this disjointness and the 35 colors of $(V, C_B)$, if $(V, C) \neq (V, C_B)$, then $(V, C)$ would have 7 colors with five lines per color. Suppose that $B(a, b) \neq B(i, j)$ and $C(ab) = C(ij)$. The orbit of $ij$ under $G_{ab}$ determines the lines colored $C(ab)$. The orbit of these lines under $G_a$ determines the seven colors of $(V, C)$. However, direct computation reveals that no automorphism of $(V, C)$ switches $a$ and $b$, showing $(V, C)$ is not a 2-t ec-graph. Thus $(V, C)$ is either monochromatic or derived from the BIBD.

(iii) Suppose that $G = HS \leq A(V, C)$ and $|V| = 176$. As in (i), $(V, C)$ is monochromatic.

(iv) Suppose that $G = CA_1 \leq A(V, C)$ and $|V| = 276$. As in (i), $(V, C)$ is monochromatic.

(v) Suppose that $G = S_2(q) \leq A(V, C)$, where $q = 2^{2a+1}$ and $a \geq 1$, and $|V| = q^2 + 1$. As in (i), $(V, C)$ is monochromatic (see [11,19, p. 250]).

(vi) Suppose that $G = PSL(2, q) \leq A(V, C)$, where $q$ is a power of a prime, and $|V| = q + 1$. As in (i), $(V, C)$ is monochromatic.

(vii) Suppose that $G = PSL(n + 1, q) \leq A(V, C)$, where $q$ is a power of a prime, $n \geq 2$, and $|V| = \sum_{i=0}^{n} q^i$. If $x$ is not incident with $l = B(a, b)$, as in (i), $(V, C)$ is monochromatic. Now suppose that $x$ is incident with $l$. Because $G$ is 3-t on $l$, Theorem 1 forces $l$ to be monochromatic. Thus $(V, C_B) \leq (V, C)$, where $(V, C_B)$ is derived from $PG(n, q)$. If $(V, C) = (V, C_B)$, we are done. Otherwise there must be some line $B(u, v)$ with $l \cap B(u, v) = \emptyset$ such that $C(uv) = C(ab)$. Then $v$ is not in the plane determined by $a$, $b$ and $u$, so as in (i), $(V, C)$ is monochromatic or derived from a BIBD.

(viii) Suppose that $G = PSp(2n, 2) \leq A(V, C)$, where $n \geq 3$, and $|V| = 2^{2n-1} + 2^n - 1$. As in (i), $(V, C)$ is monochromatic.

(ix) Suppose that $G = U_3(q) \leq A(V, C)$, where $q$ is a power of a prime and $q > 2$, and $|V| = q^2 + 1$. As in (i), $(V, C)$ is monochromatic (see [1,12,16]).

(x) Suppose that $G = G_2(q) \leq A(V, C)$, where $q = 3^{2a+1}$ and $q > 3$, and $|V| = q^3 + 1$. As in (i), $(V, C)$ is monochromatic (see [15]).

Table 2 summarizes the classification in Theorem 6, abbreviating "$q$ is a power of a prime" by "$q = p^n$", "monochromatic" by "mono", and "trivial" by "tr."

Theorem 6. If $(V, C)$ is a finite 2-t ec-graph and $G$ is a simple group acting doubly transitivity on $V$ with $G \leq A(V, C)$, then either $G = S_4$ and $(V, C)$ is monochromatic or trivial or else one of the following cases occurs:

(i) $G$ is a projective, unitary, or Ree group and $(V, C)$ is the 2-t ec-graph resulting from the unique BIBD on $V$ with $\lambda = 1$ determined by $G$;

(ii) $(V, C)$ is a one-factorization and $G = PSL(2, p)$ for $p = 3, 5, 7$ or 11;
Table 2

<table>
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<th>Group</th>
<th>Size of V</th>
<th>All finite 2-t ec-graphs</th>
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<td>mono, tr</td>
</tr>
<tr>
<td>$A_7$</td>
<td>15</td>
<td>mono, tr, BIBD, Ex. 5</td>
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<tr>
<td>$HS$</td>
<td>176</td>
<td>mono, tr</td>
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<tr>
<td>$C_3$</td>
<td>276</td>
<td>mono, tr</td>
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<tr>
<td>$Sz(q)$, $q = 2^{2a+1}, a \geq 1$</td>
<td>$q^2 + 1$</td>
<td>mono, tr, Ex. 3, 11</td>
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<tr>
<td>$PSL(2,q)$, $q = p^m$</td>
<td>$q + 1$</td>
<td>mono, tr, BIBD, Ex. 5, 6, 7</td>
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<tr>
<td>$PSL(n+1,q)$, $q = p^m$</td>
<td>$\sum_{i=0}^{n} q^i$</td>
<td>mono, tr, BIBD, Ex. 5, 6, 7</td>
</tr>
<tr>
<td>$PSp(2n,2)$, $n \geq 3$</td>
<td>$2^{2a+1} + 2^{2a+1}$</td>
<td>mono, tr, Ex. 9</td>
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<tr>
<td>$U_3(q)$, $q \neq 2$, $q = p^m$</td>
<td>$q^3 + 1$</td>
<td>mono, tr, BIBD, Ex. 8, 10</td>
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<tr>
<td>$^2G_2(q)$, $q &gt; 3$, $q = 3^{2a+1}$</td>
<td>$q^3 + 1$</td>
<td>mono, tr, BIBD, Ex. 8</td>
</tr>
</tbody>
</table>

(iii) $G = A_7$ and $(V, C)$ is one of the 2-t ec-graphs in Example 5;
(iv) $G = PSL(n,p)$ for $p = 2$, 3 or 5 and $(V, C)$ is a 2-t ec-graph in a families in Example 5, 6, or 7;
(v) $G = U_3(3)$ or $G = U_3(5)$ and $(V, C)$ is one of the 2-t ec-graphs in Example 8 or 10;
(vi) $G = PSp(2m,2)$ and $(V, C)$ is a 2-t ec-graph in one of the two families in Example 9;
(vii) $G = PSL(2,9)$ and $(V, C)$ is the 2-t ec-graph in Example 11.

**Proof.** Assume that $e_{ab} = 1$ for any edge $ab$ and there are distinct edges $ab$ and $xy$ with $C(ab) = C(xy)$, since otherwise $(V, C)$ is classified in Theorem 5 or is the trivial graph. From Lemma 4 $e = k_{ab}/2$, $e$ divides $|E|$, and $e \leq |V|/2$. Further, $e = |V|/2$ if $G$ is a one-factorization. The orbit of the edge $xy$ under the group $G_{ab}$ determines $e$. If $x$ and $y$ are in the same orbit of size $r$, the size of the orbit of $xy$ is $r/2$ to ensure $e_{ab} = e_{xy} = 1$. Similarly if $x$ and $y$ are in different orbits, these orbits must be the same size $r$.

(i) Suppose that $G = PSL(2,11) \leq A(V, C)$ and $|V| = 11$. The orbit of $x$ under $G_{ab}$ has three or six elements. Only $r = 6$ is possible, but $e = 1 + r/2 = 4$ does not divide $|E|$. Hence $(V, C)$ is monochromatic or trivial.
(ii) Suppose that $G = A_7 \leq A(V, C)$ and $|V| = 15$. As in (i) and Theorem 5(ii) $(V, C)$ is monochromatic or trivial, derived from a BIBD, or Example 5.
(iii) Suppose that $G = HS \leq A(V, C)$ and $|V| = 176$. As in (i) $(V, C)$ is monochromatic or trivial (see [8,17]).
(iv) Suppose that $G = C_3 \leq A(V, C)$ and $|V| = 276$. As in (i) $(V, C)$ is monochromatic or trivial (see [8]).
(v) Suppose that $G = Sz(q) \leq A(V, C)$ and $|V| = q^2 + 1$. As in Theorem 5(v) $(V, C)$ is monochromatic or trivial.
(vi) Suppose that $G = PSL(2,q)$, where $q = p^m$, for some prime $p$, and $|V| = q + 1$.
Assume that $V = PG(1,q)$. If $p = 2$, $G$ is 3-t and $|V| = 2^m + 1$ is odd,
contradicting Theorem 1. If \((V, C)\) is a one-factorization, then \(q \in \{3, 5, 7, 11\}\) by Cameron and Korchevskor [8]. Combinatorial restrictions eliminate all other options except \(q = 9\), fulfilled in Example 11. Because \(\text{PGL}(2, 9)\) is 3-t on \(\text{PG}(1, 9)\), any two such 2-t ec-graphs are isomorphic. Hence \((V, C)\) is monochromatic, trivial, a one-factorization, or Example 11.

(vii) Suppose that \(G = \text{PSL}(n + 1, q)\), where \(q = p^m\), for \(p\) a prime, \(n \geq 2\), and \(|V| = \sum_{i=0}^{m} q^i\). Assume that \(V = \text{PG}(n, q)\). Then \(G_{\alpha}\) leaves the line \(B(a, b) = B\) stable. The orbit of \(x\) under \(G_{\alpha}\) is either \(V/B\) or \(B/\{a, b\}\). Suppose first that this orbit is \(V/B\). If \(B \cap B(x, y)\) is empty, then \((V, C)\) is a one-factorization, contradicting Cameron and Korchevskor [8]. So assume that \(\{x\} = B \cap B(x, y)\). Then \(B\) has just three points because \(G\) is 3-t on each line, giving us Example 5. Finally suppose that the orbit of \(x\) under \(G_{\alpha}\) is \(B/\{a, b\}\). Hence we have a one-factorization of \(B\), and \(G\) is 3-t on \(B\). By Cameron and Korchevskor [8] and Cameron [6] \(B\) is either 4, 6, or 8. The values of 4 and 6 correspond to Examples 6 and 7. If \(|B| = 8\), then \(G\) would act on \(B\) as \(\text{PGL}(2, 7)\). However, the 3-t ec-graph on eight vertices has \(A(V, C) = \text{AGL}(3, 2)\). Thus \((V, C)\) is monochromatic, trivial, derived from a BIBD, or one of Examples 5, 6, and 7.

(viii) Suppose that \(G = \text{PSp}(2n, 2) \leq A(V, C), n \geq 3\) and \(|V| = 2^{2n-1} \pm 2^n - 1\). Arguments similar to previous ones but somewhat involved force \((V, C)\) to be monochromatic, trivial, or Example 9.

(ix) Suppose that \(G = U_{\lambda}(q) \leq A(V, C)\), where \(q\) is a power of a prime, \(q \neq 2\), and \(|V| = q^{\lambda} + 1\). As in (i) and Theorem 5(ix), \((V, C)\) is monochromatic, trivial, Example 8, or Example 10 (see [1; 10, p.14; 12; 16]).

(x) Suppose that \(G = \text{PG}_{\alpha}(q) \leq A(V, C)\), where \(q = 3^{2n+1}, q > 3\), and \(|V| = q^3 + 1\). As in Theorem 5(x), \((V, C)\) is monochromatic, trivial, or derived from a BIBD. □

In addition to the simple groups and the affine family of groups, the 2-t groups include groups containing \(\text{PGL}(2, 8)\), acting on a set of 28 unitals, whose 2-t ec-graphs are classified in Theorem 7.

**Theorem 7.** If \((V, C)\) is a 2-t ec-graph, where \(V\) is the set of unitals for \(G = \text{PG}_{\alpha}(2, 8) = \text{PG}_{\alpha}(3) \leq A(V, C)\), then

(i) \((V, C)\) is monochromatic;
(ii) \((V, C)\) is trivial;
(iii) \((V, C)\) is derived from the BIBD on \(V\) with \(\lambda = 1\);
(iv) \((V, C)\) is a one-factorization on 28 vertices;
(v) \((V, C)\) has, for any color \(c\), \(K(V, C, c) \cong \text{PGL}(2, 8)\);
(vi) \((V, C)\) is the meet of possibilities (iii) and (iv) (Example 8); or
(vii) \((V, C)\) is the join of possibilities (iii) and (iv).
Proof. For $a \neq b$ in $V$ we find all $K$ with $G_{(a,b)} \leq K \leq G$. Let $J = PGL(2,8)$, a simple normal subgroup of $G$ with $G_{(a,b)} \leq J$. The subgroups $K$ such that $G_{(a,b)} \leq K \leq J$ are $G_{(a,b)}$, $J$, $AGL(2,8)$, and $T$, the eight translations of $AGL(2,8)$ (see [10,6]). These correspond, respectively, to the possibilities (ii), (v), (iv), and (vi) in the theorem. There are at most four more subgroups of $PGL(2,8)$ whose intersections with $J$ are one of these four subgroups. Three of these potential subgroups actually exist: $PGL(2,8)$, $AFL(2,8)$ and the group $B$ for the unique BIBD with $\lambda = 1$ (see [14]). These correspond, respectively, to the possibilities (i), (vii), and (iii) in the theorem. The fourth would have index 2 in $B$. However, that would entail partitioning the six edges of each line in the BIBD into two sets of three edges, which is not 2-t.

Remark. The 2-t ec-graph in Theorem 7 (vii) is a 3-factorization of the complete graph of order 28.

2. The affine case

Let $V$ be an $n$-dimensional vector space over a finite field $F$ with $|F| = p^k$ and
\[ G = A(V, C) \leq AGL(n, p^k). \]
Now $AGL(n, p^k) \leq AGL(nk, p)$, so $A(V, C) \leq AGL(d, p)$, $d = nk$. In the non-affine case, Lemma 3 let us use only the minimal 2-transitive subgroups because all the related groups contain these minimal groups. Unfortunately, in the affine case the situation is reversed: $A(V, C)$ is a subgroup of the large group, $AGL(d, p)$. Further, as Table 3 illustrates, the often large number of non-isomorphic 2-t ec-graphs for such groups makes a complete classification infeasible. This situation contrasts with the relatively few non-affine examples in Table 3. (The non-simple group $PGL(2,8)$ accounts for most of the examples when $|V| = 28$.)

We can describe the chromomorphism subgroups in the affine case, although the case $p = 2$ is more complicated than other primes. Because all chromomorphism subgroups $K(c)$ are conjugate, when $p > 2$ we choose $c = C(-ev)$ for a non-zero vector $v$, and when $p = 2$ we choose $c = C(0v)$. For $p > 2$ Theorem 14 shows that a chromomorphism subgroup $K(C(-ev)) = K$ is the semidirect product of its subgroup $T_K$ of “translations” and its subgroup $K_0$ fixing 0. Thus the construction in Example 12 gives all 2-t ec-graphs for $p > 2$, providing a suitable classification. The “central symmetry” switching each $x$ with $-x$ plays a special role when $p > 2$, so we reserve the letter $a$ for it. When $p = 2$ Example 13 gives a family of 2-t ec-graphs besides those of Example 12, showing that the analog of Theorem 14 fails. Nevertheless, when $p = 2$ we can construct all 2-t ec-graphs from Examples 12 and 13.

<table>
<thead>
<tr>
<th>Size of $V$</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>13</th>
<th>16</th>
<th>25</th>
<th>27</th>
<th>28</th>
<th>31</th>
<th>49</th>
<th>64</th>
<th>81</th>
</tr>
</thead>
<tbody>
<tr>
<td>Non-isom.</td>
<td>6</td>
<td>4</td>
<td>8</td>
<td>10</td>
<td>10</td>
<td>$\geq$18</td>
<td>8</td>
<td>10</td>
<td>12</td>
<td>$\geq$20</td>
<td>$\geq$19</td>
<td>$\geq$28</td>
</tr>
<tr>
<td>Not affine</td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>10</td>
<td>6</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>
Definitions and notations. Let $G = A(V, C)$ be a 2-t subgroup of $AGL(d, p)$. For $v \in V$ define the translation $t_v$ by $t_v(x) = x + v$, and $T = \{ t_v : v \in V \}$. Denote the non-zero elements of $V$ by $V^\ast$. For $p > 2$ and $v \in V^\ast$ suppose that $G_{\{v, v, \ldots, v\}} \leq K \leq G$. Define $T_K = T \cap K$, $V_{T_K} = \{ w : t_w \in T_K \}$, $K_0 = \{ g \in G_0 : \exists t \in T : tg \in K \}$, and $V_K$ to be the subspace generated by $\{ g(v) : g \in K_0 \}$. For $p > 2$ define $\sigma \in AGL(d, p)$ by $\sigma(x) = -x$ for all $x \in V$. For $p = 2$, replace $G_{\{v, v, \ldots, v\}}$ with $G_{\{0, v\}}$.

**Lemma 8.** $T$ is normal in $G$, and $T_K$ is normal in $K$. Every $a \in G$ can be written uniquely as $a = tg$, where $t \in T$ and $g \in G_0$; $V_{T_K}$ is a subspace of $V$; $K_0$ is a subgroup of $G_0$ with $K_0 \leq K_0$; If $v \in V_{T_K}$, then $V_K \leq V_{T_K}$; If $g \in K_0$, then $g(V_{T_K}) = V_{T_K}$ and $g(V_K) = V_K$.

**Proof.** All of these results are well known or easily shown. □

**Example 12.** First let $p > 2$, $v \in V^\ast$, and $G$ be a 2-t subgroup of $AGL(d, p)$. By Lemma 2 if $G_{\{v, v, \ldots, v\}} \leq K_0 \leq G_0$, then $(V, C_K)$ is a 2-t ec-graph. Let $V_K$ be the subspace generated by $\{ g(v) : g \in K_0 \}$. Let $V'$ be any subspace of $V$ such that for all $g \in K_0$, $g(V') = V'$ and $T' = \{ t_w : w \in V' \}$. Then $T'K_0 = K$ is a subgroup of $G$ and $(V, C_K)$ is a 2-t ec-graph. For example, $V' = V$ and $V' = \{ 0 \}$ are always stable for any $K_0$. If $V' = V$, then $K = TK_0$ gives a regular 2-t ec-graph; that is, each color of $(V, C_K)$ is a regular graph. In general, the groups $G$ and $K_0$ determine which proper subspaces are stable. For any stable $V'$ either $V_K \leq V'$ or $V_K \cap V' = \{ 0 \}$. In the first case, the graph is related to the BIBD where $V_K$ is a block (see Figs. 4a and b).

For $p = 2$ every chromomorphism group $K(V, C(0, 0)) = K$ contains the translations $t_{u+v}$ such that $C(uv) = C(0v)$. Thus, $V_K \leq V'$. If $T' = \{ t_u : u \in V' \}$ and for all $g \in K_0$, $g(V') = V'$, the semidirect product $K = T'K_0$ is a subgroup with $G_{\{0, v\}} \leq K$. Then for any 2-t group $G$ and $v \in V^\ast$ this construction gives a 2-t ec-graph.

For $p > 2$ Theorem 14 below shows that $K = T_KK_0$, showing that Example 12 describes all such 2-t ec-graphs. Lemmas 10–13 consider the types of 2-t subgroups of $AGL(d, p)$, given in Kantor [14].

![Fig. 4. (a) $v \in V'$, (b) $v \notin V'$](image-url)
Lemma 9. Suppose that \( p > 2 \), \( G = A(V, C) \) is doubly transitive, \( G \leq AGL(d, p) \), \( v \in V^* \), \( G_{[-r, t]} \leq K \leq G \) and \( \sigma \in G \). Then \( K = T_KK_0 \).

Proof. Note that \( \sigma \in G_{[-r, t]} \leq K \), \( \sigma \) commutes with all \( g \in G_0 \) and for \( t, t_0 \in T \), \( \sigma t_0 \sigma = t \). For \( k \in K \), write \( k = t_0 g \) with \( t_0 \in T \) and \( g \in G_0 \). Then \( \sigma(t_0 g) \sigma(t_0 g)^{-1} = t \). \( k \in K \) because \( p \) is odd and \( t_0 \in K \). Thus \( g = t_0(t_0 g) \in K \), and so \( t_0 \in K \) and \( g \in G_0 \).

Lemma 10. Suppose \( G = A(V, C) \) is a doubly transitive group with \( ASL(n, q^a) \leq G \leq AGL(n, q^a) \), \( |V| = q^n \), \( n \geq 2 \) and \( p > 2 \). If \( v \in V^* \) and \( G_{[-r, t]} \leq K \leq G \), then \( K = T_KK_0 \).

Proof. If \( n \) is even, then the determinant of \( \sigma \) is 1. Thus \( \sigma \in ASL(n, q) \leq G \) and \( K = T_KK_0 \).

Let \( n \) be odd and \( \kappa \in K = K(C(-v)) \). For \( \kappa = t_g \) with \( t \in T \) and \( g \in G_0 \), we know \( g \in K_0 \). We need to show that \( t \in T_K \), for then \( g = t^{-1}(tg) \in K \cap K_0 = K_0 \). Let \( u = \kappa(0) = t(0) \) and \( v = g(u) \). The action of automorphisms on the subspace \( \langle u, v, w \rangle \) forces \( t \in T_K \).

Lemma 11. If \( G = A(V, C) \) is a doubly transitive subgroup of \( AGL(1, p^d) \) and \( p > 2 \), then \( \sigma \in G \).

Proof. If \( V \) is the field of order \( p^d \), then \( \Gamma_0 = AGL(1, p^d)_0 \) is a semidirect product of the cyclic groups \( V^* \) and \( Aut(V) \). For a generator of \( V^* \) and \( \beta \) the Frobenius map \( \beta(x) = x^{p^d} \), the elements \( a \beta^k \) of \( \Gamma_0 \) satisfy \( (a \beta^k)(a \beta^m) = a \beta^k(a \beta^m) = a \beta^{k+m} \). Because \( G \) is 2-transitive, there is \( a \beta^k \in G_0 \leq \Gamma_0 \) mapping 1 to \( a \). Now \( (a \beta^k)^j = \prod_{i=0}^{j-1}(a \beta^i(a \beta^m) \in \langle a \rangle \rangle \), which we call \( a^j \). Then \( z = \sum_{j=0}^{j-1} p^d = j \left( \frac{d-1}{p-1} \right) \) has even order in \( Z_{p^d-1} \). Since \( \sigma \) is the only element of order 2 in \( V^* \) we have \( \sigma \in \langle a^j \rangle \leq G \).

Lemma 12. Suppose that \( G = A(V, C) \) is a doubly transitive subgroup of \( AGL(d, p) \), \( p > 2 \), \( v \in V^* \), \( G_{[-r, t]} \leq K \leq G \), and \( T_K = \{ t_0 \} \). Then \( K = K_0 \).

Proof. By the conjugacy of all chromomorphism groups, for all \( a, b \in V \), \( \Gamma \cap K(C(ab)) = \{ t_0 \} \). For any edge \( ab \), the \( p^d \) translates \( \langle a + w \rangle \langle b + w \rangle \) of \( ab \) all have different colors, so the number of colors is a multiple of \( p^d \). Let \( Q_{ab} = \{ x : \exists y : C(xy) = C(ab) \} \). By Lemma 4 the number of edges per color divides \( (p^d - 1)/2 \) and so \( |Q_{ab}| \) divides \( p^d - 1 \).

Let \( S_{ab} = \sum_{x \in Q_{ab}} x \). Because \( |Q_{ab}| \) is relatively prime to \( p \), there is a unique \( c_{ab} \in V \) such that \( |Q_{ab}|c_{ab} = S_{ab} \). For \( \kappa \in K(C(ab)) \), \( Q_{\kappa(a)b} = Q_{ab} \) and so \( K(C(ab)) \) fixes \( c_{ab} \). Let \( a' = \kappa(a) \) and \( b' = \kappa(b) \), where \( \kappa(x) = x - c_{ab} \). Then \( S_{ab} = \sum_{x \in Q_{ab}} x = \sum_{x \in Q_{ab}} x - c_{ab} = S_{ab} - |Q_{ab}||c_{ab} = 0 \). Hence \( K(C(a'b')) = G_0 \), implying \( b' = a' \).

For any \( v \in V^* \) by conjugacy \( K(C(-v)) = K_0 \leq G_0 \).
Lemma 13. Suppose that $G = A(V, C)$ is a doubly transitive subgroup of $AGL(2, p)$, where $p = 5, 7, 11, 19, 23, 29$, or $59$ and $SL(2, 3) \triangleleft G_0$ or $SL(2, 5) \triangleleft G_0$. For $v \in V^*$ if $G_{[v]} \leq K = K(C(-vv)) \leq G$, then $K = T_K K_0$.

Proof. Since $d = 2$ either $T_k = \{t_0\}$, $T_k = T$, or $T_k$ is one-dimensional. If $T_k = \{t_0\}$, use Lemma 12. If $T_k = T$, every subgroup $K$ with $T \leq K$ satisfies $K = T K_0$. We split the third case into two subcases: $t_v \in T_k \leq K = K(C(-vv))$ and $t_v \not\in T_k$. Because all isomorphism subgroups are conjugate we may choose $v = (1, -1)$ in the first subcase and $v = (1, 1)$ in the second. By Dixon and Mortimer [11, p. 239], $G_0$ contains

$$
\mu = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.
$$

In either case $\mu \in G_{[-v, v]} \leq K_0$. In the first subcase we have $K = T_K K_0$, and so $K = T_K K_0$. In the second subcase we have $\sigma \in G$, so we can use Lemma 9. $\Box$

Theorem 14. If $(V, C)$ is a $2$-t ec-graph with $A(V, C) \leq AGL(d, p)$ for $p \geq 2$, $v \in V^*$ and $K = K(C(-vv))$, then $K = T_K K_0$.

Proof. We use the classification in Kantor [14] of $2$-t subgroups of $AGL(d, p)$ and the previous lemmas.

(i) For $G \leq AGL(1, p^d)$ use Lemmas 11 and 9.
(ii) For $ASL(n, p^d) \leq G$, where $n \geq 2$, use Lemma 10.
(iii) For $Sp(n, p^d) \leq G$, use Lemma 9 because $\sigma \in Sp(n, p^d)$.
(iv) For $SL(2, 3) \triangleleft G_0$ or $SL(2, 5) \triangleleft G_0$ and $|V| = p^2$, $p = 5, 7, 11, 19, 23, 29$ or $59$, use Lemma 13.
(v) By Aschbacher [3] the three remaining groups with $|V| = 3^d$, $d = 4$ or $d = 6$ contain $\sigma$, so use Lemma 9. $\Box$

The classification of $2$-t affine groups in Kantor [14], Theorem 14, and Example 12 provides a method, even if burdensome, to construct any affine $2$-t ec-graph when $p > 2$. Given $G$ a $2$-t subgroup of $AGL(d, p)$ and any $v \in V^*$, find all $K_0$ such that $G_{[v]} \leq K_0 \leq G_0$, all subgroups $T'$ of translations, and all subspaces left stable by $K = T' K_0$. From $G$, $K_0$, and $T'$ Example 12 gives any $2$-t affine ec-graph.

We turn now to the case $p = 2$ which admits the more complicated construction in Example 13.

Example 13. Let $V$ be the field of order $2^{2n}$ and $F$ the subfield of order $2^k$, where $1 < k, n$. The elements of $G = AGL(1, 2^{2n})$ can be written as $g_{a,b}$, where $g_{a,b}(x) = ax + b$ and $a \in V^*$, $b \in V$. Note that $t_v = g_{1,0}$ and $G_{\{0,1\}} = \{t_0, t_1\}$. Let $w$ generate $V^*$, $j = (2^{2n} - 1)/(2^k - 1)$ and $u = w^j$. Then $u$ generates $F^*$. Let $K = K(V, C, C(01))$ be the subgroup generated by $g_{u,w}$ and the translations $\{t_q : q \in F\} = T_K$. Then
Fig. 5. (a) Four parallel lines in the BIBD with 16 vertices and 4 vertices per line. (b) Coloring of the edges of the four lines from (a) following the construction in Example 13.

$G_{[0,1]} \leq T_K \leq K$. The elements of $K$ are of the form $g_{d,x+y}$, where $q \in F$. Thus, $K$ is not the semidirect product $T_K K_0$. If $L = T_K K_0 = \{g_{d,x}: q \in F\}$, then $|K| = |L|$. Further, $(V, C_L)$ is derived from the BIBD in which $F$ is one line. Both $(V, C_K)$ and $(V, C_L)$ are 2-t ec-graphs with the same translations in $K(C_K(ab))$ and $K(C_L(ab))$, but edges in one line of $(V, C_L)$ are split up among translates of those edges to form $(V, C_K)$. Fig. 5a and b illustrate this situation.

Example 13 complicates finding all 2-t ec-graphs when case $p = 2$. Fortunately, if $K$ satisfies $G_{[0,2]} \leq K \leq G$, where $G$ is a 2-t subgroup of $AGL(d, 2)$ and $v \in V$, then $L = T_K K_0$ satisfies $|K| = |L|$ and $K \cap L = T_K K_0$, which we call $J$. Lemma 8 ensures that $G_{[0,v]} \leq J \leq L$. Clearly, $(V, C_K)$, $(V, C_L)$ and $(V, C_J)$ are 2-t ec-graphs with $(V, C_J) \leq (V, C_K)$ and $(V, C_J) \leq (V, C_L)$. Further, $T_K = T_L = T_J$. The set of edges of one color for either $(V, C_K)$ or $(V, C_L)$ is a union of $[K_0]/[K_0]$ families of same colored edges of $(V, C_J)$. As in Example 13, the families forming one color in $(V, C_K)$ are translates of the families forming one color in $(V, C_L)$. The constructions of Example 12 and this generalization of Example 13 are the only ones possible when $p = 2$.

3. Regular edge colored graphs

As mentioned in Example 4, we can convert a metric space to an ec-graph. Regular ec-graphs correspond to metric spaces in which the configuration of distances from any point to other points is independent of the point. A geometrically interesting and more restricted family of metric spaces are those with transitive isometry groups, as defined below. We classify the 2-t ec-graphs corresponding to both of these situations as well as symmetric association schemes.

Definition. An edge colored graph $(V, C)$ is regular iff for each color $c$, the edges $ab$ such that $C(ab) = c$ form a regular graph on $V$.

Theorem 15. If $(V, C)$ is a finite regular 2-t ec-graph, then either $(V, C)$ is monochromatic; $(V, C)$ is a one-factorization; $|V| = 28$ and $A(V, C) = PGL(2, 8)$; or $|V| = p^d$, $A(V, C) \leq AGL(d, p)$, and for any edge $ab$, $T \leq K(V, C, ab)$. 


Proof. We consider three cases: $A(V, C)$ contains a simple 2-t group, it contains $P(2, 8)$, or it is a subgroup of some affine group $AGL(d, p)$. In the first case Theorem 6 shows that only monochromatic ec-graphs and one-factorizations are regular. For the second case, parts (i), (iv), (v), and (vii) of Theorem 7 list the four regular 2-t ec-graphs. Finally, for the third case suppose $A(V, C) \subseteq AGL(d, p)$ and $|V| = p^d$. First let $p > 2$. In order that $(V, C)$ be regular, the number of edges of any color must be a multiple of $p^d$. For $K = K(V, C, C(-v))$, the number of edges of the color $C(-v)$ is $|K|/|G_{(K, -v)}| = |T_k||K_0|/|G_{(K, -v)}|$. Now $(V, C_{K_0})$ is a 2-t ec-graph with $|K_0|/|G_{(K, -v)}|$ edges of color $C_K(-v)$. In Lemma 12 we showed that $|K_0|/|G_{(K, -v)}|$ is relatively prime to $p^d$. Hence $|T_k| = p^d$ and $T_k = T$. A similar argument holds when $p = 2$ once we substitute $K_0$ for $K_0$ and $G_{K_0}$ for $G_{(K, -v)}$. \[\square\]

Example 14. Let $V$ be the $d$-dimensional vector space over $Z_p$, $G = AGL(d, p)$, $ab$ any edge in $V$ and $H = TG_{(a,b)}$ (or $TG_{(a,b)}$ if $p = 2$). Now $T$ is the smallest transitive subgroup of $G$, and so $H$ is the smallest transitive subgroup containing $G_{(a,b)}$ (or $G_{(a,b)}$, if $p = 2$). Thus all regular 2-t ec-graphs $(V, C)$ satisfy $(V, C_H) \leq (V, C)$.

Remark. $T$ and $V$ are isomorphic as groups, so $(V, C_H)$ corresponds to the equidistance relation on $V$ defined in Sibley [18].

From Example 14 the regular affine 2-t ec-graphs correspond to the subgroups $K$ such that $TG_{(a,b)} \leq K \leq G$ (or $TG_{(a,b)} \leq K \leq G$, if $p = 2$). What more can we say about these graphs? The number of colors of $(V, C)$ must divide the number of colors of $(V, C_H)$ in Example 14, which is $\frac{p^d - 1}{2}$ if $p > 2$ and $p^d - 1$ if $p = 2$. Example 15 shows that all such divisors are possible. Unfortunately, Example 16 shows that there are non-isomorphic 2-t ec-graphs for some divisors.

Example 15. Let $V$ be the field of order $p^d$ and $G = AGL(1, p^d)$. The cyclic group $G_0 = V^*$ has $p^d - 1$ elements. For each divisor $j$ of $p^d - 1$ there is a unique subgroup $J_j$ of $V^*$ containing $j$ elements. If $p > 2$, assume that $j$ is an even divisor; if $p = 2$, $j$ can be any divisor. If we use $K_j = TJ_j$ in Lemma 2, then $(V, C_j)$ has $(|V| - 1)/j$ colors.

Example 16. Let $V$ be the two-dimensional vector space over $Z_3$. There are non-isomorphic regular 2-t ec-graphs $(V, C)$ and $(V, C')$ with $C(E) = C'(E) = \{0, 1, 3\}$. Call the six classes of parallel lines $B_m$, for $m \in Z_3 \cup \{\infty\}$, where the lines $y = m\lambda + c$, for $m, c \in Z_3$ are in $B_m$ and the lines $x = c$ are in $B_0$. Define

$$C(ab) = \begin{cases} 0 & \text{if } B(a, b) \in B_0 \cup B_3 \\ 1 & \text{if } B(a, b) \in B_1 \cup B_4 \\ 3 & \text{if } B(a, b) \in B_2 \cup B_3 \end{cases}$$
and

\[
C'(ab) = \begin{cases} 
0 & \text{if } B(a, b) \in B_0 \cup B_1 \\
1 & \text{if } B(a, b) \in B_1 \cup B_2 \\
3 & \text{if } B(a, b) \in B_3 \cup B_4 
\end{cases}
\]

Since \( A(V, C) \) has 4800 elements and \( A(V, C') \) has only 2400 elements \((V, C)\) and \((V, C')\) are not isomorphic. (For each color \( e \) some \( e \)-chromomorphisms switch the other two colors, so \( K(V, C, e) \) and \( K(V, C', e) \) each have index 6 in \( A(V, C) \) and \( A(V, C') \), respectively.)

Theorem 16 classifies the 2-t ec-graphs with a transitive group of isometries, a stronger condition than regular.

**Definitions.** An automorphism \( \rho \) of \((V, C)\) is an *isometry* iff \( C(ab) = C(\rho(a)\rho(b)) \) for all edges \( ab \). Denote the group of isometries by \( I(V, C) \). An ec-graph \((V, C)\) is *point color symmetric* iff \( I(V, C) \) is transitive on \( V \) and \( A(V, C) \) is transitive on the colors \( C(E) \). (See Chen and Teh [9] for more on point color symmetric graphs.)

**Theorem 16.** A 2-t ec-graph is point color symmetric iff it is a doubly transitive symmetric association scheme iff it monochromatic or regular and affine or item \((v)\) of Theorem 7.

**Proof.** A 2-t group \( A(V, C) \) is transitive on \( C(E) \). If \( I(V, C) \) is transitive, then \((V, C)\) is regular. Among the 2-t ec-graphs in Theorem 15 only those listed in this theorem have transitive isometry groups, which are thus the point color symmetric ones. By definition a symmetric association scheme is a regular ec-graph. Among the 2-t ec-graphs in Theorem 15 only those listed in this theorem satisfy the definition of a symmetric association scheme. (See Bannai and Ito [5, p. 52].) \( \square \)

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**References**