1.4 Axiomatic Systems, Models, and Metamathematics

In Section 1.3 we showed the logical need for undefined terms in an axiomatic system, which ignores what those terms mean. However, people depend on meaning and intuition to create and understand mathematics. A string stretched taut between its ends provides a strong intuition of a line, but such an image is too imprecise for mathematics. Mathematical models provide an explicit link between intuitions and undefined terms. The usual analytic model of Euclidean plane geometry is the set $\mathbf{R}^2 = \{(x, y) : x, y \in \mathbf{R}\}$, where a *point* is interpreted as an ordered pair of numbers (x, y) and a *line* is interpreted as the points that satisfy an appropriate first degree equation $ax + by + c = 0$. High school students spend considerable time learning how this algebraic model matches geometric intuition and axioms. In making a model, we are free to interpret the undefined terms in any way we want, provided that all the axioms hold under our interpretation. Note that the axioms are not by themselves true; a context is needed to give meaning to the axioms in order for them to be true or false.

Definition 1.4.1 A *model* of an axiomatic system is a set of objects together with interpretations of all the undefined terms of the axiomatic system such that all the axioms are true in the set using the interpretations.