

Let us show that the axioms are consistent. We will use the method of models. The idea is to interpret the undefined terms of the axioms in a way that makes the axioms true. Usually we choose the model to be a concrete one.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Clearly, we need to choose some models for the undefined terms. Let us choose the model to be a set of points. We will use the following axioms: (1) For any two points  $A$  and  $B$ , there is a unique line containing both of them. (2) If two lines intersect, then they intersect in exactly one point. (3) There are at least three points not on the same line. (4) There are at least two lines that do not intersect. (5) If a line  $l$  and a point  $P$  are not on the same line, then there is exactly one line through  $P$  that does not intersect  $l$ .

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### 1.4 AXIOMATIC SYSTEMS, MODELS, AND METAMATHEMATICS

In Section 1.3 we showed the logical need for undefined terms in an axiomatic system, which ignores what those terms mean. However, people depend on meaning and intuition to create and understand mathematics. A string stretched taut between its ends provides a strong intuition of a line, but such an image is too imprecise for mathematics. Mathematical models provide an explicit link between intuitions and undefined terms. The usual analytic model of Euclidean plane geometry is the set  $\mathbf{R}^2 = \{(x, y) : x, y \in \mathbf{R}\}$ , where a *point* is interpreted as an ordered pair of numbers  $(x, y)$  and a *line* is interpreted as the points that satisfy an appropriate first degree equation  $ax + by + c = 0$ . High school students spend considerable time learning how this algebraic model matches geometric intuition and axioms. In making a model, we are free to interpret the undefined terms in any way we want, provided that all the axioms hold under our interpretation. Note that the axioms are not by themselves true; a context is needed to give meaning to the axioms in order for them to be true or false.

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**Definition 1.4.1** A *model* of an axiomatic system is a set of objects together with interpretations of all the undefined terms of the axiomatic system such that all the axioms are true in the set using the interpretations.