3.4 Area and Hyperbolic Designs

Areas in hyperbolic geometry do not have easily remembered formulas, such as $A = \frac{1}{2}bh$ for a Euclidean triangle. In place of that, Theorem 3.4.5 asserts that the area of a triangle is proportional to the *defect* of the triangle, or the amount by which its angle

sum is less than 180°. Without formulas for areas, we need to make explicit what we mean by area. Whether in Euclidean, hyperbolic or another geometry, area satisfies four axioms.

i) Area is a nonnegative real number.
ii) Congruent sets have the same area.
iii) The area of a disjoint union of sets is the sum of their areas.
iv) The area of any point or line segment is zero.

Example 1 Show that a triangle and the associated Saccheri quadrilateral from Theorem 3.3.3 have the same area.

Solution. From case 1 of Theorem 3.3.3, $\triangle AFD \cong \triangle BGD$ and $\triangle AEF \cong \triangle CEH$, and quadrilateral $BDEC$ is congruent to itself (Fig. 3.19). By axiom (ii), the pieces of $\triangle ABC$ have the same areas as the corresponding pieces of the Saccheri quadrilateral. But we cannot yet use axiom (iii), for these pieces are not disjoint. However, the intersections of the pieces are just line segments and so have zero area by axiom (iv). Thus the areas of the triangle and its associated Saccheri quadrilateral are equal. Cases 2 and 3 are left as exercises. ●

Theorem 3.4.1 generalizes Example 1 and justifies the definition of equivalent polygons. The notion of equivalent polygons applies in many geometries. For example, W. Bolyai and P. Gerwien showed that if two Euclidean polygons have the same area, they are equivalent. Theorem 3.4.2 enables us to show that two triangles have the same area by comparing their Saccheri quadrilaterals.

Theorem 3.4.1 If a set S is the union of a finite number of sets $A_1, A_2, \ldots, A_n$ and the intersection of any two sets A_i and A_j is a finite number of line segments, then the area of S is the sum of the areas of $A_1, A_2, \ldots, A_n$.

Proof. Problem 1. ■

Definition 3.4.1 Two polygons A and B are *equivalent* iff there are finitely many triangles $A_1, A_2, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ such that A is the union of the A_i; B is the union of the B_i; for each i, $A_i \cong B_i$; and the intersection of any A_i and A_j (or B_i and B_j) is at most a line segment.

Theorem 3.4.2 If polygons A and B are equivalent and polygons B and C are equivalent, then A is equivalent to C.

Proof. Divide A and B into families of congruent triangles, A_i and B_i. Again, divide B and C into families of congruent triangles, B'_j and C'_j, where B_i and B'_j can be different (Fig. 3.25). We subdivide the triangles B_i and B'_j into smaller triangles B_{ijk} so that we can reassemble them into either A or C. The intersection B_{ij} of the convex sets B_i and B'_j is convex by Problem 8(e) of Section 1.3. Because B_{ij} is a convex polygon, it can be subdivided into triangles B_{ijk}. Then the union of all these small triangles B_{ijk} must give B and, by the assumptions of equivalence, can be rearranged to form both A

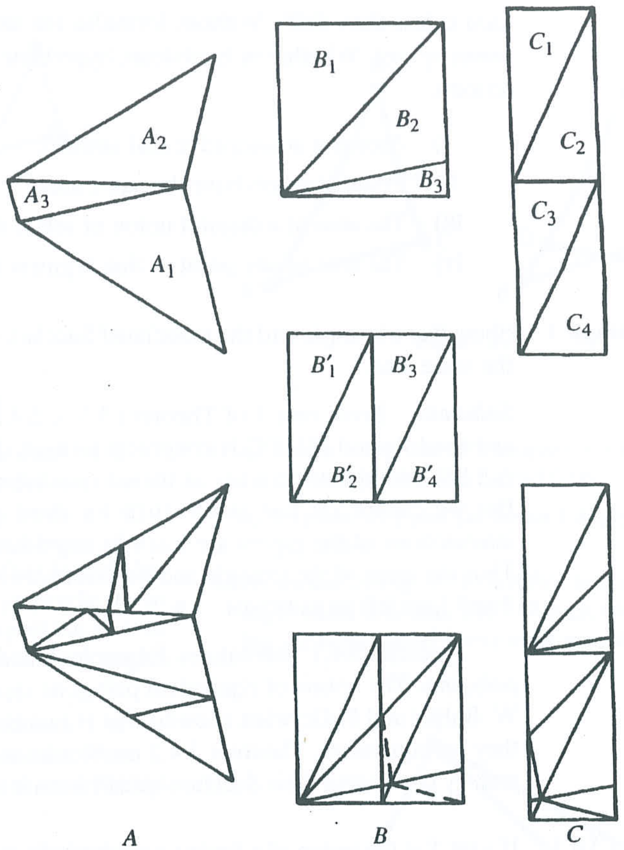

Figure 3.25

and C. This result shows that A and C can be divided into triangles A_{ijk} and C_{ijk} to make A and C equivalent. ∎

Theorem 3.4.3 Two Saccheri quadrilaterals with congruent summits and congruent summit angles are congruent and so have congruent sides and bases.

Proof. Let $ABCD$ and $EFGH$ be two Saccheri quadrilaterals, with $\overline{CD} \cong \overline{GH}$, $\angle ADC \cong \angle EHG$, and bases $\overline{AB}$ and $\overline{EF}$. Showing that $\overline{BC} \cong \overline{FG}$ and $\overline{AB} \cong \overline{EF}$ is sufficient to show that the quadrilaterals are congruent, as their corresponding angles are already congruent. Suppose, for a contradiction, that their sides are not congruent. WLOG, say, $BC > FG$ (Fig. 3.26). Hence there is a point B' on $\overline{BC}$ such that $\overline{B'C} \cong \overline{FG}$. Similarly, there is an A' on $\overline{AD}$ such that $\overline{A'D} \cong \overline{EH}$. From the midpoints P and Q of the summits $\overline{CD}$ and $\overline{GH}$ draw the line segments $\overline{A'P}$, $\overline{PB'}$, $\overline{EQ}$, and $\overline{QF}$. In Problem 2 you are asked to complete the proof of this theorem. ∎

Definition 3.4.2 The *defect* of a triangle is the difference between 180° and the angle sum of the triangle.

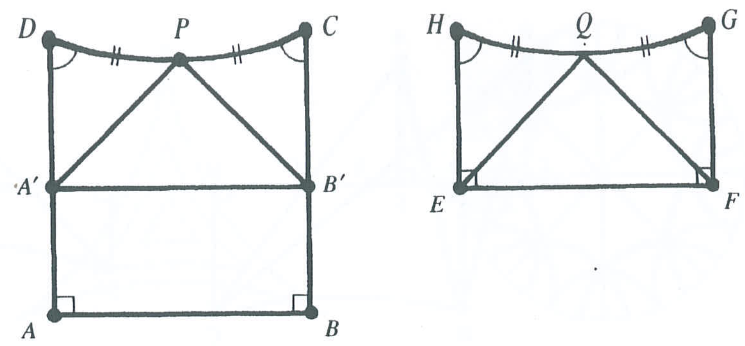

Figure 3.26

Theorem 3.4.4 Triangles with the same defect have the same area.

Proof. *Case 1* Suppose that the two triangles $\triangle ABC$ and $\triangle A'B'C'$ have the same defect and, further, that they have a pair of corresponding sides, $\overline{BC} \cong \overline{B'C'}$, congruent. We construct the associated Saccheri quadrilaterals for each triangle, $GHCB$ and $G'H'C'B'$ (Fig. 3.27). Recall that each triangle is equivalent to its Saccheri quadrilateral (Example 1) and that the angle sums of the summit angles of these two Saccheri quadrilaterals equal the angle sum of the corresponding triangles (the proof of Theorem 3.3.3). These sums are equal, so the summit angles of the Saccheri quadrilaterals must be congruent. By Theorem 3.4.3 the Saccheri quadrilaterals are congruent and so

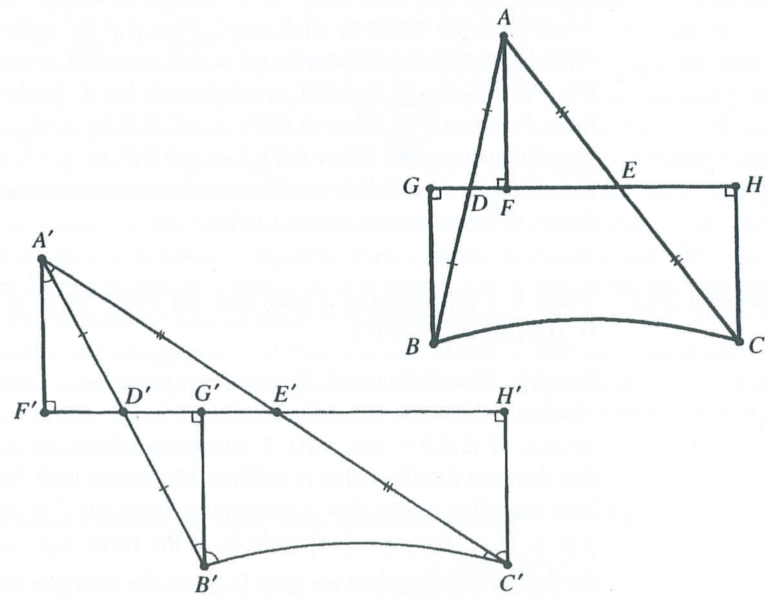

Figure 3.27

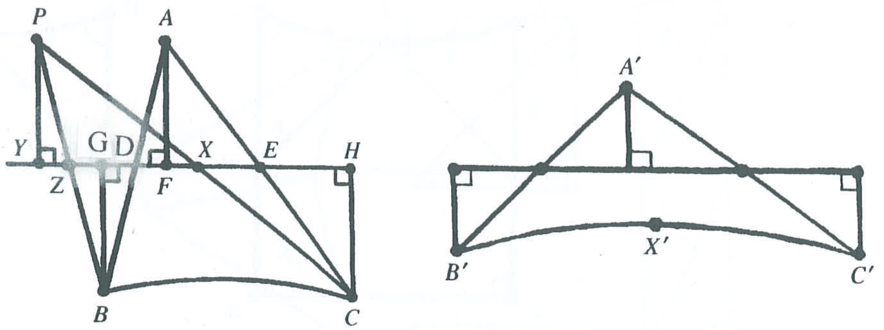

Figure 3.28

are equivalent. Then Theorem 3.4.2 shows the two triangles are equivalent and so have the same area.

Case 2 For the general case, we suppose only that $\triangle ABC$ and $\triangle A'B'C'$ have the same defect. We construct a third triangle with one side congruent to one side of the first triangle and another side congruent to one side of the other triangle and with the same defect as the given triangles. Then we use case 1 twice to conclude that $\triangle ABC$ and $\triangle A'B'C'$ are both equivalent to this third triangle. Hence we need only to construct this triangle and prove that it satisfies the needed properties (Fig. 3.28). Let $\overline{B'C'}$ be the longest of the six sides of the two triangles and let X' be the midpoint of $\overline{B'C'}$. There is a point X on line $\overleftrightarrow{GH}$, which includes the base of the Saccheri quadrilateral $\overline{GH}$, such that $\overline{XC} \cong \overline{X'C'}$. (We can find such an X because $\overline{EC}$ is shorter than $\overline{X'C'}$.) Construct P on $\overleftrightarrow{XC}$ with X the midpoint of P and C. Then $\triangle PBC$ is the candidate for the third triangle. Next, show that the intersection Z of $\overline{PB}$ with $\overleftrightarrow{GH}$ is the midpoint of $\overline{PB}$ so that $\triangle PBC$ has $GHCB$ for its Saccheri quadrilateral. Let Y be the point where the perpendicular from P meets $\overleftrightarrow{GH}$. Then $\triangle PYX \cong \triangle CHX$ by AAS, implying that $\overline{PY} \cong \overline{CH}$, which is congruent to $\overline{BG}$. Thus $\triangle PYZ \cong \triangle BGZ$, by AAS, and Z is indeed the midpoint of $\overline{PB}$. Finally, as $\triangle PBC$ has the same Saccheri quadrilateral as $\triangle ABC$, it has the same defect. We can now use case 1 twice. ■

Theorem 3.4.5 There is a real number k such that, for every triangle $\triangle ABC$, the area of $\triangle ABC$ is $k \cdot$ (Defect of $\triangle ABC$).

Proof. The defect and the area are functions of the triangle. First, Problem 9 of Section 3.3 shows the defect function to be additive: For P between B and C, the defects of $\triangle ABP$ and $\triangle ACP$ add to the defect of $\triangle ABC$. Area axiom (iii) shows that the area function also is additive. Moreover, both functions are continuous. A result from calculus states that a continuous function f that is additive [that is, it satisfies $f(x+y) = f(x) + f(y)$] must be of the form $f(x) = cx$, for some constant c. Both the defect function and the area function for triangles have this form, so one must be a multiple of the other: The area of $\triangle ABC$ is $k \cdot$ (Defect of $\triangle ABC$). ■

Figure 3.29

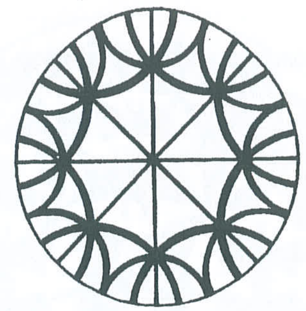

Figure 3.30

Theorem 3.4.6 The area of a convex polygon is proportional to the defect of that polygon.

Proof. See Problem 5(b). ■

The tie between the area of a polygon and its defect leads to curious possibilities. In Euclidean geometry, all equilateral triangles must have 60° angles. In hyperbolic geometry, however, the angles of an equilateral triangle, though congruent, must be less than 60°. Furthermore, as the sides of the triangle lengthen continuously, this angle measure must decrease continuously, by Theorem 3.4.5 (Fig. 3.29). Thus some unique length for this angle measures exactly 45°, so we can fit eight equilateral triangles around a point. We can extend this pattern to cover the entire plane, as partially illustrated in Fig. 3.30 for the Poincaré model. We can create a corresponding pattern with seven or more equilateral triangles around a point. Similarly, we can create patterns with five or more "squares," where a "square" is a figure with four congruent sides and four congruent angles. With the help of the geometer H. S. M. Coxeter, the Dutch artist M. C. Escher laboriously created several imaginative patterns, building on hyperbolic designs like that shown in Fig. 3.30. More recently, Douglas Dunham [4] has used geometry and computer graphics to create more varied patterns relatively quickly (Figs. 3.31 and 3.32). From one copy of the repeating motif the computer constructs congruent copies to fill out the plane. Of course, the computer needs to be programmed to "draw" in hyperbolic geometry instead of Euclidean geometry. Inversions, which we discuss in Section 4.6, provide one key to drawing congruent shapes for the Poincaré model. We also need to compute distances between points in this model. Let A and B be any two hyperbolic points. The hyperbolic line on these two points intersects the boundary of the Poincaré model in the two omega points of that line, say, Ω and Λ. Poincaré found the hyperbolic distance between A and B in terms of the Euclidean distances among the four points A, B, Ω, and Λ (Fig. 3.33). (The formula in Definition 3.4.3 involves the cross-ratio, which we discuss in Chapter 6.)

Definition 3.4.3 In the Poincaré model, the *hyperbolic distance* between A and B is

$$d_H(A, B) = c \cdot \left|\log\left(\frac{A\Omega/B\Omega}{A\Lambda/B\Lambda}\right)\right|,$$

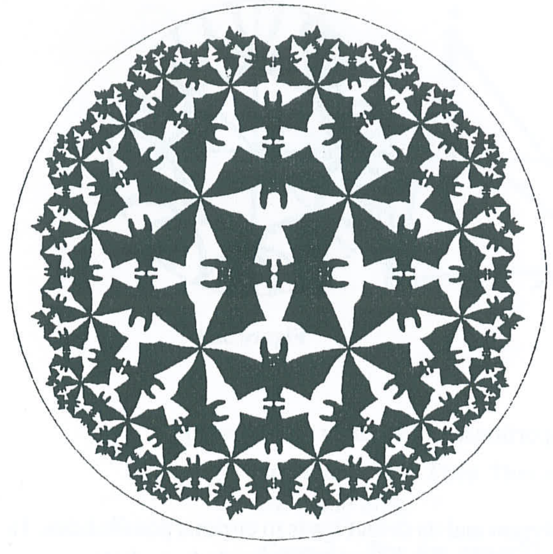

Figure 3.31

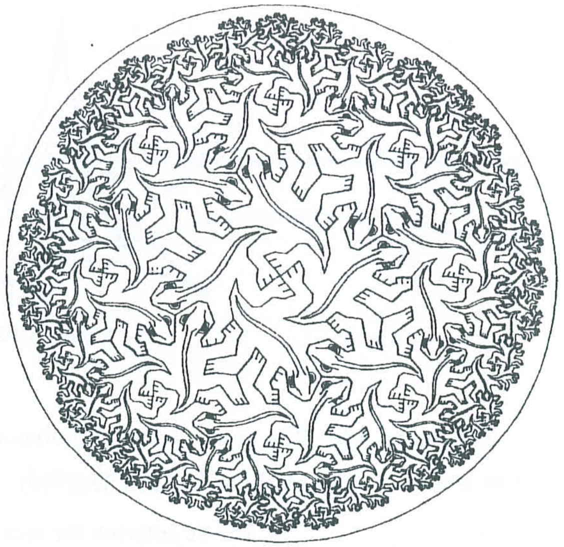

Figure 3.32

where XY is the Euclidean distance, c is some constant, and Ω and Λ are the two omega points of line $\overleftrightarrow{AB}$.

Example 2 Verify that neighboring points P_i in Fig. 3.34 are equally spaced. The x-coordinates of the points are $P_0 = 0$, $P_1 = \frac{1}{3}$, $P_2 = \frac{3}{5}$, $P_3 = \frac{7}{9}$, $P_4 = \frac{15}{17}$, $P_5 = \frac{31}{33}$, $P_{-i} = -P_i$, $\Omega = -1$, and $\Lambda = 1$.

Solution. The Euclidean distances between these points are simply the differences of their x-coordinates. Then $(P_0\Omega/P_1\Omega) \div (P_0\Lambda/P_1\Lambda) = (1/(4/3)) \div (1/(2/3)) = 1/2$. Similarly, $(P_1\Omega/P_2\Omega) \div (P_1\Lambda/P_2\Lambda) = ((4/3)/(8/5)) \div ((2/3)/(2/5)) = 1/2$. All of the corresponding quotients equal 1/2 or 2. In turn, the absolute values of their logarithms are all the same. Hence, whatever the constant c is, the distances will all be equal. ●

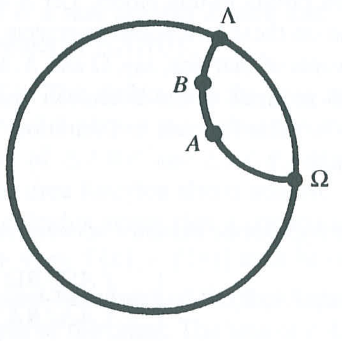

Figure 3.33

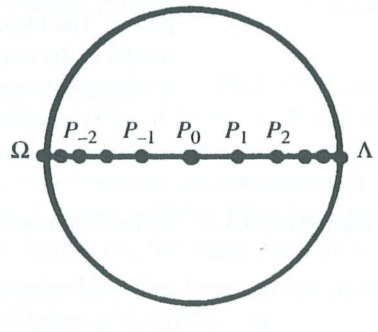

Figure 3.34

PROBLEMS FOR SECTION 3.4

1. Prove Theorem 3.4.1.
2. Complete the proof of Theorem 3.4.3 as follows.
   a) Prove that $\triangle A'DP \cong \triangle EHQ$.
   b) Prove that $\triangle A'PB' \cong \triangle EQF$.
   c) Prove that quadrilateral $AA'B'B$ would have to have four right angles, which is a contradiction. This contradiction forces $\overline{BC} \cong \overline{FG}$ and $\overline{AD} \cong \overline{EH}$.
   d) Prove that the bases, $\overline{AB}$ and $\overline{EF}$, must be congruent.
3. Prove that there is some real number K such that the area of every triangle is less than K. (The smallest such K is the area of a "triangle" that has three omega points for its vertices.)
4. Use Fig. 3.35 and the theorems of this section to prove that hyperbolic triangles having the same height and congruent bases don't necessarily have the same area. What happens to the area of $\triangle AB_iB_{i+1}$ as $i \to \infty$?

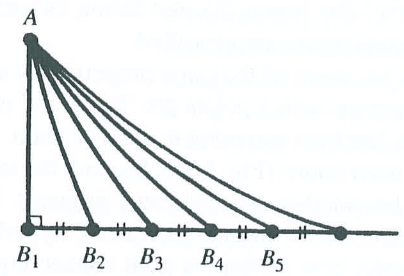

Figure 3.35

5. Recall that the angle sum of a convex Euclidean polygon with n sides is $(n-2)180°$.
   a) Prove by induction that the angle sum of a convex hyperbolic polygon with n sides is less than $(n-2)180°$.
   b) The defect of a convex hyperbolic polygon with n sides is the difference between $(n-2)180°$ and its angle sum. Prove that the area of the polygon is proportional to its defect.
6. Generalize Problem 3 to show polygons with n sides have a largest area. Find the relationship between the least upper bound K_n of the areas of polygons with n sides and $K = K_3$, the least upper bound for the areas of triangles.
7. We can construct the inner circle of equilateral triangles shown in Fig. 3.30 with the help of Fig. 3.36.

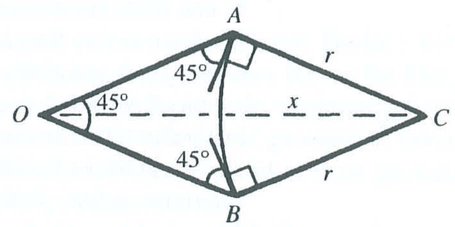

Figure 3.36

   a) Construct the unit circle, the x-axis, the y-axis, and $y = \pm x$.
   b) The remaining side of each of the eight triangles is an arc of a circle orthogonal to the unit circle. Explain why these circles all have their centers on the lines forming angles of $22\frac{1}{2}° + k \cdot 45°$ with the x-axis. Explain why these circles all have their centers the same distance, $x = OC$, from the origin and have the same, as yet unknown, radius r.
   c) Explain why, in Fig. 3.36, OA and OB equal r.
   d) Use the Poincaré model to explain why, in Fig. 3.36, $x^2 = 1 + r^2$.
   e) The law of cosines gives a second equation in x and r, namely, $x^2 = r^2 + r^2 - 2r^2 \cos 135°$. Find x and r.
   f) Finish constructing the eight equilateral triangles from part (a).
8. **a)** Verify that $C : (x - \frac{5}{3})^2 + y^2 = (\frac{4}{3})^2$ is orthogonal to the unit circle.
   b) Find the intersections Ω and Λ of circle C with the unit circle. Verify that $P = (\frac{1}{3}, 0)$, $Q = (\frac{1}{2}, \frac{\sqrt{15}}{6})$, and $R = (\frac{2}{5}, \frac{\sqrt{39}}{15})$ are on C. Assume $c = 1$ and find $d_H(P, Q)$, $d_H(Q, R)$, and $d_H(P, R)$. Why would we expect the sum of the two smaller distances to equal the larger distance? Verify that they do.
9. Show the following properties for d_H in the Poincaré model.

a) $d_H(A, B) = d_H(B, A) \geq 0$ and $d_H(A, B) = 0$ iff $A = B$.

b) If B is between A and C on the diameter AC, then $d_H(A, B) + d_H(B, C) = d_H(A, C)$.

10. a) Find the general pattern for the x-coordinates of the points P_i in Example 2.

b) Use part (a) to show that $d_H(P_i, P_{i+1}) = d_H(P_0, P_1)$.

3.5 Spherical and Single Elliptic Geometries

In one sense, mathematicians have studied the geometry of the sphere for millennia. However, before Bernhard Riemann in 1854 no one had thought of spherical geometry as a separate geometry, but only as properties of a Euclidean figure. The characteristic axiom of spherical geometry is that every two lines (great circles) always intersect in two points. (See Section 1.6.)

To retain the familiar notion of Euclidean and hyperbolic geometries that two points determine a line, Felix Klein in 1874 saw the need to modify spherical geometry. The usual way to do so was to identify opposite points on the sphere as the same point and study this "collapsed" geometry, which Klein called *single elliptic geometry*. Thus the characteristic axiom of single elliptic geometry is that every two distinct lines intersect in only one point. (Klein called spherical geometry *double elliptic geometry* because lines intersect in two points.) Spherical and single elliptic geometries share many theorems in common, such as the angle sum of a triangle is greater than 180°. In addition, single elliptic geometry possesses some unusual features worth noting. We can represent single elliptic geometry as the half of a sphere facing us (Fig. 3.37) so long as we remember that a line (or curve) that leaves the part facing us immediately reappears directly opposite because opposite points are identified.

A line in either of these geometries has many of the same properties as a circle in Euclidean geometry. First, we can't determine which points are "between" two points because there are two ways to go along a line from one point to another point. Note that we can use two points to "separate" two other points (Fig. 3.38). Second, the total length of a line is finite. A single elliptic line has another, more unusual property: It doesn't separate the whole geometry into two parts, unlike lines in Euclidean, hyperbolic, and spherical geometries. Figure 3.37 indicates how to draw a path connecting any two points not on a given line so that the path does not cross that line.

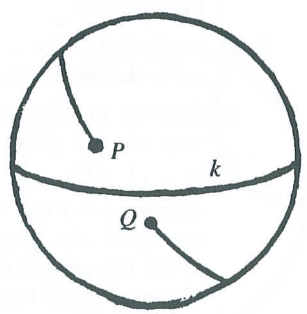

Figure 3.37 In the single elliptic geometry there is a path from P to Q that does not intersect k.

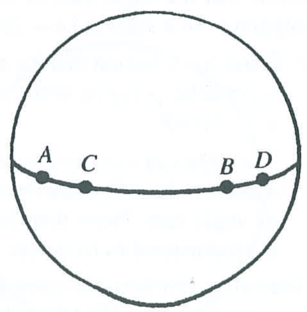

Figure 3.38 A and B separate C and D.

In certain ways, Euclidean geometry is intermediate between spherical and single elliptic geometries on the one hand and hyperbolic geometry on the other hand. For example, in Euclidean geometry, the angle sum of a triangle always adds to 180°. As we know in hyperbolic geometry, the corresponding sum falls short of 180° in proportion to the area of the triangle. In spherical and single elliptic geometries, this sum is always more than 180° and the excess is proportional to the area of a triangle. (Theorem 1.6.3 shows this condition for Euclidean spheres.) Indeed, in these geometries triangles can have three obtuse angles, so the sum can approach 540°.

In our development of hyperbolic geometry we assumed that Euclid's first 28 propositions hold, for they used only Euclid's first four postulates, but not the fifth postulate. Many of these propositions, including two of the triangle congruence theorems (SAS and SSS), continue to hold in spherical and single elliptic geometries.* However, most of the propositions after I-15, including AAS, do not hold in these geometries, even though they do not depend on the fifth postulate.

Figure 3.39 illustrates Euclid's approach to showing, as I-16 states, that in any triangle an exterior angle, such as $\angle BCD$, is larger than either of the other two interior angles, $\angle ABC$ and $\angle BAC$. From the midpoint E of $\overline{BC}$, Euclid extended AE to F so that $\overline{EF} \cong \overline{EA}$. Then by SAS $\triangle ECF \cong \triangle EBA$. He then concluded that $\angle BCD$ is larger than $\angle ECF$, which is congruent to the interior angle $\angle EBA$. Figure 3.39 supports this conclusion, but the similar situation shown in Fig. 3.40 for single elliptic geometry reveals that the conclusion depends on implicit assumptions. In Fig. 3.40, the part of $\overleftrightarrow{AE}$ that looks like segment $\overline{AE}$ covers more than half the length of the line. Hence the corresponding part of $\overleftrightarrow{EF}$ overlaps this apparent segment. Euclid implicitly assumed that lines extend infinitely in each direction. Postulate 2 only says, "to produce a finite straight line continuously in a straight line." The overlapping "segments" $\overline{AE}$ and $\overline{EF}$ in Fig. 3.40 satisfy the letter and, within reason, the spirit of postulate 2. Nevertheless, I-16 is false here because $\angle BCD$ can be smaller than $\angle ECF$.

Exercise 1 Draw the figure in spherical geometry corresponding to the situation depicted in Fig. 3.40.

* For SAS to hold we need to assume that a side is the shortest part of a geodesic.

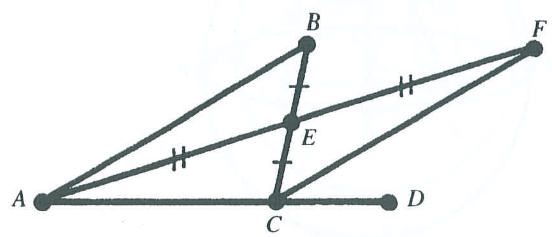

Figure 3.39 Euclid's diagram for proposition I-16.

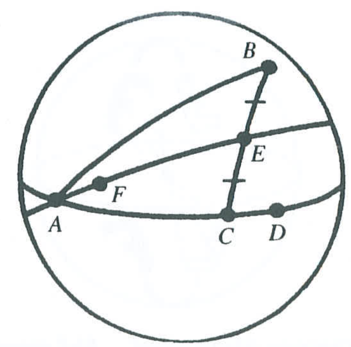

Figure 3.40 The diagram from Fig. 3.39 in single elliptic geometry.

a) $d_H(A, B) = d_H(B, A) \geq 0$ and $d_H(A, B) = 0$ iff $A = B$.

b) If B is between A and C on the diameter AC, then $d_H(A, B) + d_H(B, C) = d_H(A, C)$.

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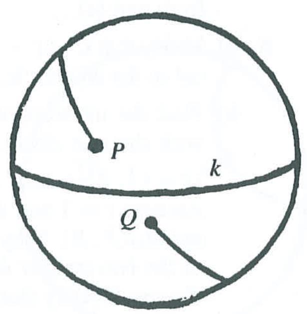

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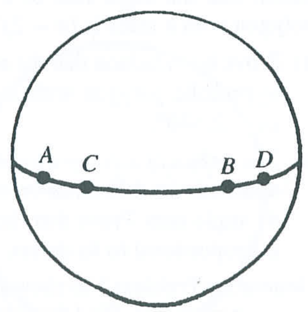

Figure 3.38 A and B separate C and D.